

Null controllability of Grushin-type operators in dimension two

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Abstract

We study the null controllability of the parabolic equation associated with the Grushin-type operator $A = \partial_x^2 + |x|^{2\gamma} \partial_y^2$, ($\gamma > 0$), in the rectangle $\Omega = (-1, 1) \times (0, 1)$, under an additive control supported in the strip $\omega = (a, b) \times (0, 1)$, ($0 < a, b < 1$). We prove that the equation is null controllable in any positive time for $\gamma < 1$, and that it fails to be so for $\gamma > 1$. In the transition regime $\gamma = 1$, we show that both behaviors live together: a positive minimal time is required for null controllability. Our approach is based on the fact that, thanks to the particular geometric configuration, null controllability is equivalent to the observability of the Fourier components of the solution of the adjoint system uniformly with respect to the frequency.

Key words: null controllability, degenerate parabolic equations, Carleman estimates

AMS subject classifications: 35K65, 93B05, 93B07, 34B25

1 Introduction

1.1 Main result

We consider the Grushin-type equation

$$\begin{cases} \partial_t f - \partial_x^2 f - |x|^{2\gamma} \partial_y^2 f = u(t, x, y) 1_\omega(x, y) & (t, x, y) \in (0, \infty) \times \Omega, \\ f(t, x, y) = 0 & (t, x, y) \in (0, \infty) \times \partial\Omega, \end{cases} \quad (1)$$

where $\Omega := (-1, 1) \times (0, 1)$, $\omega \subset \Omega$, and $\gamma > 0$. Problem (1) is a linear control system in which

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- the state is f ,
- the control u is supported in the subset ω .

It is a degenerate parabolic equation, since the coefficient of $\partial_y^2 f$ vanishes on the line $\{x = 0\}$. We will investigate the null controllability of (1).

Definition 1 (Null controllability). *Let $T > 0$. System (1) is null controllable in time T if, for every $f_0 \in L^2(\Omega)$, there exists $u \in L^2((0, T) \times \Omega; \mathbb{R})$ such that the solution of*

$$\begin{cases} \partial_t f - \partial_x^2 f - |x|^{2\gamma} \partial_y^2 f = u(t, x, y) 1_\omega(x, y) & (t, x, y) \in (0, T) \times \Omega, \\ f(t, x, y) = 0 & (t, x, y) \in (0, T) \times \partial\Omega, \\ f(0, x, y) = f_0(x, y) & (x, y) \in \Omega, \end{cases} \quad (2)$$

satisfies $f(T, \cdot, \cdot) = 0$.

System (1) is null controllable if there exists $T > 0$ such that it is null controllable in time T .

The main result of this paper is the following one.

Theorem 1. *Let $\omega = (a, b) \times (0, 1)$, where $0 < a < b < 1$.*

1. *If $\gamma \in (0, 1)$, then system (1) is null controllable in any time $T > 0$.*
2. *If $\gamma = 1$, then there exists $T^* > 0$ such that*
 - *for every $T > T^*$ system (1) is null controllable in time T ,*
 - *for every $T < T^*$ system (1) is not null controllable in time T .*
3. *If $\gamma > 1$, then (1) is not null controllable.*

By duality, the null controllability of (1) is equivalent to an observability inequality for the adjoint system

$$\begin{cases} \partial_t g - \partial_x^2 g - |x|^{2\gamma} \partial_y^2 g = 0 & (t, x, y) \in (0, \infty) \times \Omega, \\ g(t, x, y) = 0 & (t, x, y) \in (0, \infty) \times \partial\Omega. \end{cases} \quad (3)$$

Definition 2 (Observability). *Let $T > 0$. System (3) is observable in ω in time T if there exists $C > 0$ such that, for every $g_0 \in L^2(\Omega)$, the solution of*

$$\begin{cases} \partial_t g - \partial_x^2 g - |x|^{2\gamma} \partial_y^2 g = 0 & (t, x, y) \in (0, T) \times \Omega, \\ g(t, x, y) = 0 & (t, x, y) \in (0, T) \times \partial\Omega, \\ g(0, x, y) = g_0(x, y) & (x, y) \in \Omega, \end{cases} \quad (4)$$

satisfies

$$\int_{\Omega} |g(T, x, y)|^2 dx dy \leq C \int_0^T \int_{\omega} |g(t, x, y)|^2 dx dy dt.$$

System (3) is observable in ω if there exists $T > 0$ such that it is observable in ω in time T .

Theorem 2. *Let $\omega = (a, b) \times (0, 1)$, where $0 < a < b < 1$.*

1. *If $\gamma \in (0, 1)$, then system (4) is observable in ω in any time $T > 0$.*
2. *If $\gamma = 1$, then there exists $T^* > 0$ such that*
 - *for every $T > T^*$ system (4) is observable in ω in time T ,*
 - *for every $T < T^*$ system (4) is not observable in ω in time T .*
3. *If $\gamma > 1$, then system (4) is not observable in ω .*

1.2 Motivation and bibliographical comments

1.2.1 Null controllability of the heat equation

The null and approximate controllability of the heat equation are essentially well understood subjects for both linear and semilinear equations, and for bounded or unbounded domains (see, for instance, [12], [14], [16], [17], [18], [22], [26], [27], [30], [33], [34], [36], [37]). Let us summarize one of the existing main results. Consider the linear heat equation

$$\begin{cases} \partial_t f - \Delta f = u(t, x)1_\omega(x) & (t, x) \in (0, T) \times \Omega, \\ f = 0 & \text{on } (0, T) \times \partial\Omega, \\ f(0, x) = f_0(x) & x \in \Omega, \end{cases} \quad (5)$$

where Ω is an open subset of \mathbb{R}^d , $d \in \mathbb{N}^*$, and ω is a subset of Ω . The following theorem is due, for the case $d = 1$, to H. Fattorini and D. Russell [15, Theorem 3.3], and, for $d \geq 2$, to O. Imanuvilov [24], [25] (see also the book [20] by A. Fursikov and O. Imanuvilov) and G. Lebeau and L. Robbiano [27].

Theorem 3. *Let us assume that Ω is bounded, of class C^2 and connected, $T > 0$, and ω is a nonempty open subset of Ω . Then the control system (5) is null controllable in time T .*

So, the heat equation on a smooth bounded domain is null controllable

- in arbitrarily small time;
- with an arbitrarily small control support ω .

It is natural to ask whether null controllability also holds for degenerate parabolic equations such as (1). Let us compare the known results for the heat equation with the results proved in this article. The first difference concerns the geometry of Ω and ω : a more restrictive configuration is assumed in Theorem 1 than in Theorem 3. The second difference concerns the structure of the controllability results. Indeed, while the heat equation is null controllable in arbitrarily small time, the same result holds for the Grushin equation only when degeneracy is not too strong (i.e. $\gamma \in (0, 1)$). On the contrary, when degeneracy is too strong (i.e. $\gamma > 1$), null controllability does not hold any more. Of special interest is the transition regime ($\gamma = 1$), where the ‘classical’ Grushin operator appears: here, both behaviors live together, and a positive minimal time is required for the null controllability.

1.2.2 Boundary-degenerate parabolic equations

The null controllability of parabolic equations degenerating on the boundary of the domain in one space dimension is well-understood, much less so in higher dimension. Given $0 < a < b < 1$ and $\alpha > 0$, let us consider the 1D equation

$$\partial_t w + \partial_x(x^\alpha \partial_x w) = u(t, x)1_{(a, b)}(x), \quad (t, x) \in (0, \infty) \times (0, 1),$$

with suitable boundary conditions. Then, it can be proved that null controllability holds if and only if $\alpha \in (0, 2)$ (see [8, 9]), while, for $\alpha \geq 2$, the best result one can show is “regional null controllability”(see [7]), which consists in controlling the solution within the domain of influence of the control. Several extensions of

the above results are available in one space dimension, see [1, 31] for equations in divergence form, [6, 5] for nondivergence form operators, and [4, 19] for cascade systems. Fewer results are available for multidimensional problems, mainly in the case of two dimensional parabolic operators which simply degenerate in the normal direction to the boundary of the space domain, see [10]. As in the above references, also for the Grushin equation null controllability holds if and only if the degeneracy is not too strong ($\gamma \in (0, 1]$).

1.2.3 Parabolic equations degenerating inside the domain

In [32], the authors study linearized Crocco type equations

$$\begin{cases} \partial_t f + \partial_x f - \partial_{vv} f = u(t, x, v) 1_\omega(x, v) & (t, x, v) \in (0, T) \times (0, L) \times (0, 1), \\ f(t, x, 0) = f(t, x, 1) = 0 & (t, x) \in (0, T) \times (0, L), \\ f(t, 0, v) = f(t, L, v) & (t, v) \in (0, T) \times (0, 1). \end{cases}$$

For a given open subset ω of $(0, L) \times (0, 1)$, they prove regional null controllability. Notice that, in the above equation, diffusion (in v) and transport (in x) are decoupled.

In [3], the authors study the Kolmogorov equation

$$\partial_t f + v \partial_x f - \partial_{vv} f = u(t, x, v) 1_\omega(x, v), \quad (x, v) \in (0, 1)^2, \quad (6)$$

with periodic type boundary conditions. They prove null controllability in arbitrarily small time, when the control region ω is a strip, parallel to the x -axis. We note that the above Kolmogorov equation degenerates on the whole space domain, unlike Grushin's equation. However, differently from the linearized Crocco equation, transport (in x at speed v) and diffusion (in v) are coupled. This is why the null controllability results are also different for these equations.

1.2.4 Unique continuation and approximate controllability

In this paper we will not directly address approximate controllability, which is another interesting problem in control theory. It is well-known that, for evolution equations, approximate controllability can be equivalently formulated as unique continuation (see [35]). The unique continuation problem for the Grushin-type operator

$$A = \partial_x^2 + |x|^{2\gamma} \partial_y^2$$

has been widely investigated. In particular, in [21] (see also the references therein) unique continuation is proved for every $\gamma > 0$ and every open set ω .

1.2.5 Null controllability and hypoellipticity

It could be interesting to analyze the connections between null controllability and hypoellipticity. We recall that a linear differential operator P with C^∞ coefficients in an open set $\Omega \subset \mathbb{R}^n$ is called hypoelliptic if, for every distribution u in Ω , we have

$$\text{sing supp } u = \text{sing supp } Pu,$$

that is, u must be a C^∞ function in every open set where so is Pu . The following sufficient condition (which is also essentially necessary) for hypoellipticity is due to Hörmander (see [23]).

Theorem 4. *Let P be a second order differential operator of the form*

$$P = \sum_{j=1}^r X_j^2 + X_0 + c,$$

where X_0, \dots, X_r denote first order homogeneous differential operators in an open set $\Omega \subset \mathbb{R}^n$ with C^∞ coefficients, and $c \in C^\infty(\Omega)$. Assume that there exists n operators among

$$X_{j_1}, [X_{j_1}, X_{j_2}], [X_{j_1}, [X_{j_2}, X_{j_3}]], \dots, [X_{j_1}, [X_{j_2}, [X_{j_3}, [\dots, X_{j_k}]\dots]]],$$

where $j_i \in \{0, 1, \dots, r\}$, which are linearly independent at any given point in Ω . Then, P is hypoelliptic.

Hörmander's condition is satisfied by the Grushin operator $A = \partial_x^2 + |x|^{2\gamma} \partial_y^2$ for every $\gamma \in \mathbb{N}^*$ (for other values of γ , the coefficients are not C^∞). Indeed, set

$$X_1(x, y) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2(x, y) := \begin{pmatrix} 0 \\ x^\gamma \end{pmatrix}.$$

Then,

$$[X_1, X_2](x, y) = \begin{pmatrix} 0 \\ \gamma x^{\gamma-1} \end{pmatrix}, \quad [X_1, [X_1, X_2]](x, y) = \begin{pmatrix} 0 \\ \gamma(\gamma-1)x^{\gamma-2} \end{pmatrix}, \dots$$

Thus, if $\gamma = 1$, Hörmander's condition is satisfied with X_1 and $[X_1, X_2]$. In general, if $\gamma \geq 1$, γ iterated Lie brackets are required.

Theorem 1 emphasizes that hypoellipticity is not sufficient for null controllability: Grushin's operator is hypoelliptic, but null controllability holds only when $\gamma = 1$.

The situation is similar for the Kolmogorov equation (6), where

$$X_0(x, v) := \begin{pmatrix} v \\ 0 \end{pmatrix}, \quad X_1(x, v) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [X_1, X_2](x, v) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Here again, null controllability holds and the first iterated Lie bracket is sufficient to satisfy Hörmander's condition.

A general result which relates null controllability to the number of iterated Lie brackets that are necessary to satisfy Hörmander's condition would be very interesting, but remains—for the time being—a challenging open problem.

1.3 Structure of the article

Section 2 is devoted to the proof of Theorem 2. In Subsection 2.1 we recall useful results about the well-posedness of Grushin's equation. In Subsection 2.2 we justify the Fourier decomposition of the solution to the adjoint system, which is needed for the proof of our main result. In Subsection 2.3 we present the strategy for the proof of Theorem 2, which relies on uniform observability estimates with respect to Fourier frequencies. In Subsection 2.4 we prove a preliminary result, related to the dissipation rate of the Fourier components of the solution to the Grushin equation. In Subsection 2.5, we prove the positive statements of Theorem 2, thanks to an appropriate Carleman inequality. In Subsection 2.6, we show the negative statements of Theorem 2, thanks to appropriate test functions to falsify the observability inequality. Then, in Section 2.7, we complete the proof of Theorem 2. Finally, in Section 3, we present several open problems and perspectives.

2 Proof of Theorem 2

2.1 Well posedness of the Cauchy-problem

Let $H := L^2(\Omega; \mathbb{R})$, denote with $\langle \cdot, \cdot \rangle$ the scalar product in H and by $\| \cdot \|_H := \langle \cdot, \cdot \rangle^{1/2}$ its norm. Define the scalar product

$$(f, g) := \int_{\Omega} (f_x g_x + |x|^{2\gamma} f_y g_y) dx dy \quad (7)$$

for every f, g in $C_0^\infty(\Omega)$, and set $V = \overline{C_0^\infty(\Omega)}^{|\cdot|_V}$, where $|f|_V := (f, f)^{1/2}$.

Observe that $H_0^1(\Omega) \subset V \subset H$, thus V is dense in H . We define the bilinear form a on V by

$$a(f, g) = -(f, g) \quad \forall f, g \in V. \quad (8)$$

Moreover, set

$$D(A) = \{f \in V : \exists c > 0 \text{ such that } |a(f, h)| \leq c \|h\|_H \forall h \in V\}, \quad (9)$$

$$\langle Af, h \rangle = a(f, h) \quad \forall h \in V. \quad (10)$$

Then, we can apply a result by Lions [29] (see also Theorem 1.18 in [35]) and conclude that $(A, D(A))$ generates an analytic semigroup $S(t)$ of contractions on H . Note that A is selfadjoint on H , and (10) implies that

$$Af = \partial_x^2 f + |x|^{2\gamma} \partial_y^2 f \quad \text{almost everywhere in } \Omega.$$

So, system (2) can be recast in the form

$$\begin{cases} f'(t) = Af(t) + u(t) & t \in [0, T], \\ f(0) = f_0, \end{cases} \quad (11)$$

where $T > 0$, $u \in L^2(0, T; H)$ and $f_0 \in H$.

Let us now recall the definition of weak solutions to (11).

Definition 3 (Weak solution). *Let $T > 0$, $u \in L^2(0, T; H)$ and $f_0 \in H$. A function $f \in C([0, T]; H)$ is a weak solution of (11) if for every $h \in D(A)$ the function $\langle f(t), h \rangle$ is absolutely continuous on $[0, T]$ and for a.e. $t \in [0, T]$*

$$\frac{d}{dt} \langle f(t), h \rangle = \langle f(t), Ah \rangle + \langle u(t), h \rangle. \quad (12)$$

Note that, as showed in [28], condition (12) is equivalent to the definition of solution by transposition, that is,

$$\begin{aligned} & \int_{\Omega} [f(t^*, x, y) \varphi(t^*, x, y) - f_0(x, y) \varphi(0, x, y)] dx dy \\ &= \int_0^{t^*} \int_{\Omega} \{f (\partial_t \varphi + \partial_x^2 \varphi + |x|^{2\gamma} \partial_y^2 \varphi) + u 1_{\omega} \varphi\} dx dy dt \end{aligned}$$

for every $\varphi \in C^2([0, T] \times \Omega; \mathbb{R})$ and $t^* \in (0, T)$.

Let us recall that, for every $T > 0$ and $u \in L^2(0, T; H)$, the mild solution $f \in C([0, T]; H)$ of (11) is defined as

$$f(t) = S(t)f_0 + \int_0^t S(t-s)u(s)ds, \quad t \in [0, T]. \quad (13)$$

From [2], we have that the mild solution to (11) is also the unique weak solution in the sense of Definition 3. The following existence and uniqueness result follows.

Proposition 1. *For every $f_0 \in H$, $T > 0$ and $u \in L^2(0, T; H)$, there exists a unique weak solution of the Cauchy problem (11). This solution satisfies*

$$\|f(t)\|_H \leq \|f_0\|_H + \|u\|_{L^2(0, T; H)} \quad \forall t \in [0, T]. \quad (14)$$

Moreover, $f(t, \cdot) \in D(A)$ for a.e. $t \in (0, T)$.

Proof: Inequality (14) follows from (13). Moreover, since $S(\cdot)$ is analytic, $f(t) \in D(A)$ for a.e. $t \in (0, T)$. \square

2.2 Fourier decomposition

Let us consider the solution of (4) in the sense of Definition 3, that is, the solution of system (11) with $u = 0$. The function g belongs to $C([0, T]; L^2(\Omega))$, so $y \mapsto g(t, x, y)$ belongs to $L^2(0, 1)$ for a.e. $(t, x) \in (0, T) \times (-1, 1)$, thus it can be developed in Fourier series in y

$$g(t, x, y) = \sum_{n \in \mathbb{N}^*} g_n(t, x) \varphi_n(y), \quad (15)$$

where

$$\varphi_n(y) := \sqrt{2} \sin(n\pi y) \quad \forall n \in \mathbb{N}^*$$

and

$$g_n(t, x) := \int_0^1 g(t, x, y) \varphi_n(y) dy \quad \forall n \in \mathbb{N}^*. \quad (16)$$

Proposition 2. *For every $n \geq 1$, $g_n(t, x)$ is the unique weak solution of*

$$\begin{cases} \partial_t g_n - \partial_x^2 g_n + (n\pi)^2 |x|^{2\gamma} g_n = 0 & (t, x) \in (0, T) \times (-1, 1), \\ g_n(t, \pm 1) = 0 & t \in (0, T), \\ g_n(0, x) = g_{0,n}(x) & x \in (-1, 1). \end{cases} \quad (17)$$

For the proof we need the following characterization of the elements of V . We denote by $L_\gamma^2(\Omega)$ the space of all the square-integrable functions with respect to the measure $d\mu = |x|^{2\gamma} dx dy$.

Lemma 1. *For every $g \in V$ there exist $\partial_x g \in L^2(\Omega)$, $\partial_y g \in L_\gamma^2(\Omega)$ such that*

$$\begin{aligned} \int_\Omega (g(x, y) \partial_x \phi(x, y) + |x|^{2\gamma} g(x, y) \partial_y \phi(x, y)) dx dy \\ = - \int_\Omega (\partial_x g(x, y) + |x|^{2\gamma} \partial_y g(x, y)) \phi(x, y) dx dy \end{aligned} \quad (18)$$

for every $\phi \in C_0^\infty(\Omega)$.

Proof: Let $g \in V$, and consider a sequence $(g^n)_{n \geq 1}$ in $C_0^\infty(\Omega)$ such that $g^n \rightarrow g$ in V , that is

$$\int_{\Omega} [(g^n - g)_x^2 + |x|^{2\gamma}(g^n - g)_y^2] dx dy \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Thus, $(\partial_x g^n)_{n \geq 1}$ is a Cauchy sequence in $L^2(\Omega)$, and $(\partial_y g^n)_{n \geq 1}$ is a Cauchy sequence in $L^2(\Omega, |x|^{2\gamma} dx dy)$, so there exist $h \in L^2(\Omega)$ and $k \in L^2(\Omega, |x|^{2\gamma} dx dy)$ such that $\partial_x g^n \rightarrow h$ in $L^2(\Omega)$ and $\partial_y g^n \rightarrow k$ in $L^2(\Omega, |x|^{2\gamma} dx dy)$. Hence,

$$\begin{aligned} \int_{\Omega} (g^n \partial_x \phi + |x|^{2\gamma} g^n \partial_y \phi) dx dy &= - \int_{\Omega} (\partial_x g^n \phi + |x|^{2\gamma} \partial_y g^n \phi) dx dy \\ \downarrow &\quad \downarrow \\ \int_{\Omega} (g \partial_x \phi + |x|^{2\gamma} g \partial_y \phi) dx dy &= - \int_{\Omega} (h \phi + |x|^{2\gamma} k \phi) dx dy \end{aligned}$$

as $n \rightarrow +\infty$. This yields the conclusion with $\partial_x g = h$ and $\partial_y g = k$. \square

For any $n \geq 1$, system (17) is a first order Cauchy problem, that admits a unique weak solution

$$\tilde{g}_n \in C^1((0, T); L^2(-1, 1)) \cap C([0, T]; H_0^1(-1, 1)) \cap L^2(0, T; H^2(-1, 1))$$

which satisfies

$$\begin{aligned} \frac{d}{dt} \left(\int_{-1}^1 \tilde{g}_n(t, x) \psi(x) dx \right) \\ + \int_{-1}^1 \left[\tilde{g}_{n,x}(t, x) \psi_x(x) + (n\pi)^2 |x|^{2\gamma} \tilde{g}_n(t, x) \psi(x) \right] dx = 0 \end{aligned} \quad (19)$$

for every $\psi \in H_0^1(-1, 1)$.

Proof of Proposition 2: In order to verify that the n th Fourier coefficient of g , defined by (16), satisfies system (17), observe that

$$g_n(0, \cdot) = g_{0,n}(\cdot), \quad g_n(t, \pm 1) = 0 \quad \forall t \in (0, T)$$

and

$$g_n(t, x) \in C^1((0, T); L^2(-1, 1)) \cap C([0, T]; H_0^1(-1, 1)).$$

Thus, it is sufficient to prove that g_n fulfills condition (19). Indeed, using the identity (16), for all $\psi \in H_0^1(-1, 1)$,

$$\begin{aligned} \frac{d}{dt} \left(\int_{-1}^1 g_n \psi dx \right) + \int_{-1}^1 (g_{n,x} \psi_x + (n\pi)^2 |x|^{2\gamma} g_n \psi) dx \\ = \int_{-1}^1 \int_0^1 \{ g_t \varphi_n \psi + g_x \varphi_n \psi_x + (n\pi)^2 |x|^{2\gamma} g \varphi_n \psi \} dy dx. \end{aligned} \quad (20)$$

On the other hand, choosing $h(x, y) = \psi(x) \varphi_n(y) \in V$ in (12),

$$\begin{aligned} 0 &= \int_0^1 \int_{-1}^1 (g_t - Ag) \psi \varphi_n dx dy \\ &= \int_0^1 \int_{-1}^1 g_t \psi \varphi_n dx dy + \int_0^1 \int_{-1}^1 (g_x \psi_x \varphi_n + |x|^{2\gamma} g_y \psi \varphi_{n,y}) dx dy \\ &= \int_0^1 \int_{-1}^1 g_t \psi \varphi_n dx dy + \int_0^1 \int_{-1}^1 (g_x \psi_x \varphi_n + (n\pi)^2 |x|^{2\gamma} g \psi \varphi_n) dx dy, \end{aligned} \quad (21)$$

where (in the last identity) we have used Lemma 1. Combining (20) and (21) completes the proof. \square

2.3 Strategy for the proof of Theorem 2

Let g be the solution of (4). Then, g can be represented as in (15), and we emphasize that, for a.e. $t \in (0, T)$, and for every $-1 \leq a_1 < b_1 \leq 1$,

$$\int_{(a_1, b_1) \times (0, 1)} |g(t, x, y)|^2 dx dy = \sum_{n=1}^{\infty} \int_{a_1}^{b_1} |g_n(t, x)|^2 dx$$

(Bessel-Parseval equality). Thus, in order to prove Theorem 2, it is sufficient to study the observability of system (17) uniformly with respect to $n \in \mathbb{N}^*$.

Definition 4 (Uniform observability). *Let $0 < a < b < 1$ and $T > 0$. System (17) is observable in (a, b) in time T uniformly with respect to $n \in \mathbb{N}^*$ if there exists $C > 0$ such that, for every $n \in \mathbb{N}^*$, $g_{0,n} \in L^2(-1, 1)$, the solution of (17) satisfies*

$$\int_{-1}^1 |g_n(T, x)|^2 dx \leq C \int_0^T \int_a^b |g_n(t, x)|^2 dx.$$

System (17) is observable in (a, b) uniformly with respect to $n \in \mathbb{N}^$ if there exists $T > 0$ such that it is observable in (a, b) in time T uniformly with respect to $n \in \mathbb{N}^*$.*

Theorem 2 is a consequence of the following statement.

Theorem 5. *We assume $0 < a < b < 1$.*

1. *If $\gamma \in (0, 1)$, then system (17) is observable in (a, b) in any time $T > 0$ uniformly with respect to $n \in \mathbb{N}^*$.*
2. *If $\gamma = 1$, there exists $T^* > 0$ such that*
 - *for every $T > T^*$, system (17) is observable in (a, b) in time T uniformly with respect to $n \in \mathbb{N}^*$,*
 - *for every $T < T^*$, system (17) is not observable in (a, b) in time T uniformly with respect to $n \in \mathbb{N}^*$.*
3. *If $\gamma > 1$, system (17) is not observable in (a, b) uniformly with respect to $n \in \mathbb{N}^*$.*

The strategy of the proof for the positive statements of Theorem 5 is standard and relies on two key ingredients:

- an explicit decay rate for the solutions of (17),
- a favorable estimate for the observability constant associated to the equation (17) and the observation domain (a, b) .

This strategy has already been used in [11], [3]. The proof of the negative statements of Theorem 5 relies on the use of appropriate test functions that falsify uniform observability.

Let us recall that explicit bounds on the observability constant of the heat equation with a potential are already known.

Theorem 6. *Let $-1 < a < b < 1$. There exists $c > 0$ such that, for every $T > 0$, $\alpha, \beta \in L^\infty((0, T) \times (-1, 1))$, $g_0 \in L^2(-1, 1)$, the solution of*

$$\begin{cases} \partial_t g - \partial_x^2 g + \beta \partial_x g + \alpha g = 0 & (t, x) \in [0, T] \times (-1, 1), \\ g(t, \pm 1) = 0 & t \in [0, T], \\ g(0, x) = g_0(x) & x \in (-1, 1), \end{cases}$$

satisfies

$$\int_{-1}^1 |g(T, x)|^2 dx \leq e^{cH(T, \|\alpha\|_\infty, \|\beta\|_\infty)} \int_0^T \int_a^b |g(t, x)|^2 dx dt,$$

where $H(T, A, B) := 1 + \frac{1}{T} + TA + A^{2/3} + (1 + T)B^2$.

For the proof of this result, we refer to [17, Theorem 1.3] in the case $\beta \equiv 0$ and to [12, Theorem 2.3] in the case $\beta \neq 0$. The optimality of the power $2/3$ of A in $H(T, A, B)$ has been proved in [13].

The positive statement of Theorem 5 may be seen as an improvement of the above estimate (relatively to the asymptotic behavior when $n \rightarrow +\infty$), in the particular case of equation (17).

2.4 Dissipation speed

Let us introduce, for every $n \in \mathbb{N}^*$, $\gamma > 0$, the operator $A_{n,\gamma}$ defined by

$$D(A_{n,\gamma}) := H^2 \cap H_0^1(-1, 1; \mathbb{R}), \quad A_{n,\gamma} \varphi := -\varphi'' + (n\pi)^2 |x|^{2\gamma} \varphi. \quad (22)$$

The smallest eigenvalue of $A_{n,\gamma}$ is given by

$$\lambda_{n,\gamma} = \min \left\{ \frac{\int_{-1}^1 [v'(x)^2 + (n\pi)^2 |x|^{2\gamma} v(x)^2] dx}{\int_{-1}^1 v(x)^2 dx}, v \in H_0^1(-1, 1) \right\}. \quad (23)$$

We are interested in the asymptotic behavior (as $n \rightarrow +\infty$) of $\lambda_{n,\gamma}$, which quantifies the dissipation speed of the solution of (17).

Thanks to a simple heuristic computation, one may expect that, for every $\gamma > 0$, $\lambda_{n,\gamma}$ behaves like $C(\gamma)n^{\frac{2}{1+\gamma}}$. Indeed, if we consider the eigenvector $v_{n,\gamma}$

$$\begin{cases} -v_{n,\gamma}''(x) + (n\pi)^2 |x|^{2\gamma} v_{n,\gamma}(x) = \lambda_{n,\gamma} v_{n,\gamma}(x) & x \in (-1, 1), \\ v_{n,\gamma}(\pm 1) = 0, \end{cases} \quad (24)$$

and the change of variable $y := lx$, $l = (n\pi)^{1/(1+\gamma)}$, $v_{n,\gamma}(x) = \psi_{n,\gamma}(y)$, we get

$$\begin{cases} -\psi_{n,\gamma}''(y) + |y|^{2\gamma} \psi_{n,\gamma}(y) = \lambda_{n,\gamma} (n\pi)^{\frac{-2}{1+\gamma}} \psi_{n,\gamma}(y) & y \in (-l, l), \\ \psi_{n,\gamma}(\pm l) = 0. \end{cases}$$

In order to prove two results related to this conjecture, we need the following lemma.

Lemma 2. *Problem (24) admits a unique positive solution with norm one. Moreover, $v_{n,\gamma}$ is even.*

Proof: Since (24) is a Sturm-Liouville problem, it is well-known that its first eigenvalue is simple, and the associated eigenfunction has no zeros. Thus, we can choose $v_{n,\gamma}$ to be strictly positive everywhere. Moreover, by normalization, we can find a unique positive solution satisfying the condition $\|v_{n,\gamma}\|_{L^2(-1,1)} = 1$. Finally, $v_{n,\gamma}$ is even. Indeed, if not so, let us consider the function $w(x) = v_{n,\gamma}(|x|)$. Then, w still belongs to $H_0^1(-1,1)$, it is a weak solution of (24) and it does not increase the functional in (23), i.e.

$$\frac{\int_{-1}^1 [w'(x)^2 + (n\pi)^2 |x|^{2\gamma} w(x)^2] dx}{\int_{-1}^1 w(x)^2 dx} \leq \frac{\int_{-1}^1 [v'_{n,\gamma}(x)^2 + (n\pi)^2 |x|^{2\gamma} v_{n,\gamma}(x)^2] dx}{\int_{-1}^1 v_{n,\gamma}(x)^2 dx}.$$

The coefficients of the equation in (24) being regular, we deduce that w is a classical solution of (24). Since $\lambda_{n,\gamma}$ is simple, it follows $v_{n,\gamma}(x) = v_{n,\gamma}(|x|)$. \square

The following result turns out to be the key point of the proof of Theorem 5.

Proposition 3. 1. For every $\gamma \in (0, 1]$, there exists $c_* = c_*(\gamma) > 0$ such that

$$\lambda_{n,\gamma} \geq c_* n^{\frac{2}{1+\gamma}} \quad \forall n \in \mathbb{N}^*.$$

2. For every $\gamma > 0$, there exists $c^* = c^*(\gamma) > 0$ such that

$$\lambda_{n,\gamma} \leq c^* n^{\frac{2}{1+\gamma}} \quad \forall n \in \mathbb{N}^*.$$

Proof: First, taking $\gamma \in (0, 1]$, let us prove the first part of the conclusion. Thanks to Lemma 2, we have

$$\lambda_{n,\gamma} = \min \left\{ \frac{\int_0^1 [v'(x)^2 + (n\pi)^2 x^{2\gamma} v(x)^2] dx}{\int_0^1 v(x)^2 dx}; v \in H_0^1(-1, 1) \right\}.$$

Thus, our goal is to prove the existence of $c_0 = c_0(\gamma) > 0$ such that

$$\int_0^1 [v'(x)^2 + (n\pi)^2 x^{2\gamma} v(x)^2] dx \geq c_0 (n\pi)^{\frac{2}{1+\gamma}} \int_0^1 v(x)^2 dx \quad \forall v \in H_0^1(-1, 1),$$

or, equivalently, the existence of $c_1 = c_1(\gamma) > 0$ such that

$$\int_0^1 v(x)^2 dx \leq c_1 \int_0^1 \left[\alpha_n^2 v'(x)^2 + \left(\frac{x}{\alpha_n} \right)^{2\gamma} v(x)^2 \right] dx \quad \forall v \in H_0^1(-1, 1), \quad (25)$$

where $\alpha_n := (n\pi)^{\frac{-1}{1+\gamma}}$. First, let us emphasize that

$$\int_{\alpha_n}^1 v(x)^2 dx \leq \int_{\alpha_n}^1 \left(\frac{x}{\alpha_n} \right)^{2\gamma} v(x)^2 dx.$$

Thus, in order to prove (25), it is sufficient to find $c_2 = c_2(\gamma) > 0$ such that

$$c_2 \int_0^{\alpha_n} v(x)^2 dx \leq \int_0^1 \left[\alpha_n^2 v'(x)^2 + \left(\frac{x}{\alpha_n} \right)^{2\gamma} v(x)^2 \right] dx \quad \forall v \in H_0^1(-1, 1). \quad (26)$$

First step: Let us prove that, for all $v \in H_0^1(-1, 1)$,

$$\gamma \int_0^{\alpha_n} v(x)^2 dx \leq \alpha_n v(\alpha_n)^2 + \int_0^1 \left[\alpha_n^2 v'(x)^2 + \left(\frac{x}{\alpha_n} \right)^{2\gamma} v(x)^2 \right] dx. \quad (27)$$

Let $v \in H_0^1(-1, 1)$. We have

$$\begin{aligned} \int_0^1 \left[\alpha_n^2 v'(x)^2 + \left(\frac{x}{\alpha_n} \right)^{2\gamma} v(x)^2 \right] dx &\geq \int_0^{\alpha_n} \left[\alpha_n^2 v'(x)^2 + \left(\frac{x}{\alpha_n} \right)^{2\gamma} v(x)^2 \right] dx \\ &\geq -2 \int_0^{\alpha_n} \alpha_n v'(x) \left(\frac{x}{\alpha_n} \right)^\gamma v(x) dx = -\alpha_n v(\alpha_n)^2 + \gamma \int_0^{\alpha_n} \left(\frac{\alpha_n}{x} \right)^{1-\gamma} v(x)^2 dx \\ &\geq -\alpha_n v(\alpha_n)^2 + \gamma \int_0^{\alpha_n} v(x)^2 dx \end{aligned}$$

because $1 - \gamma \geq 0$. This proves inequality (27).

Second step: Let us prove the existence of $c_3 = c_3(\gamma) > 0$ such that

$$\alpha_n v(\alpha_n)^2 \leq c_3 \int_0^1 \left[\alpha_n^2 v'(x)^2 + \left(\frac{x}{\alpha_n} \right)^{2\gamma} v(x)^2 \right] dx \quad \forall v \in H_0^1(-1, 1). \quad (28)$$

Let $v \in H_0^1(-1, 1)$. Since

$$v(\alpha_n) = v(x) + \int_x^{\alpha_n} v'(s) ds \quad \forall x \in (0, \alpha_n),$$

we have

$$\int_0^{\alpha_n} x^{2\gamma} v(\alpha_n)^2 dx \leq 2 \int_0^{\alpha_n} x^{2\gamma} v(x)^2 dx + 2 \int_0^{\alpha_n} x^{2\gamma} (\alpha_n - x) dx \int_0^{\alpha_n} v'(s)^2 ds,$$

which implies

$$\frac{\alpha_n^{2\gamma+1}}{2\gamma+1} v(\alpha_n)^2 \leq 2 \int_0^{\alpha_n} x^{2\gamma} v(x)^2 dx + 2 \left(\alpha_n \frac{\alpha_n^{2\gamma+1}}{2\gamma+1} - \frac{\alpha_n^{2\gamma+2}}{2\gamma+2} \right) \int_0^{\alpha_n} v'(x)^2 dx.$$

Multiplying both sides by $(2\gamma+1)\alpha_n^{-2\gamma}$, we deduce

$$\alpha_n v(\alpha_n)^2 \leq 2(2\gamma+1) \int_0^{\alpha_n} \left(\frac{x}{\alpha_n} \right)^{2\gamma} v(x)^2 dx + \frac{1}{\gamma+1} \int_0^{\alpha_n} \alpha_n^2 v'(x)^2 dx.$$

Hence, (28) holds with $c_3(\gamma) = 2(2\gamma+1)$. Combining (27) and (28) gives (26) with $c_2 = \gamma/(1+c_3)$.

Now, let $\gamma > 0$ and let us prove the second statement of Proposition 3. For every $k > 1$ we consider the function $\varphi_k(x) := (1 - k|x|)^+$, that belongs to $H_0^1(-1, 1)$. Easy computations show that

$$\int_{-1}^1 \varphi_k(x)^2 dx = \frac{2}{3k}, \quad \int_{-1}^1 \varphi_k'(x)^2 dx = 2k, \quad \int_{-1}^1 |x|^{2\gamma} \varphi_k(x)^2 dx = 2c(\gamma)k^{-1-2\gamma},$$

where

$$c(\gamma) := \left(\frac{1}{2\gamma+1} - \frac{1}{\gamma+1} + \frac{1}{2\gamma+3} \right).$$

Thus, $\lambda_{n,\gamma} \leq f_{n,\gamma}(k) := 3[k^2 + (\pi n)^2 c(\gamma)k^{-2\gamma}]$ for all $k > 1$. Since $f_{n,\gamma}$ attains its minimum at $\bar{k} = \tilde{c}(\gamma)n^{\frac{1}{\gamma+1}}$, we have $\lambda_{n,\gamma} \leq f_{n,\gamma}(\bar{k}) = C(\gamma)n^{\frac{2}{\gamma+1}}$. \square

2.5 Proof of the positive statements of Theorem 5

The goal of this section is the proof of the following results:

- if $\gamma \in (0, 1)$ and $T > 0$, then system (17) is observable in time T uniformly with respect to n ;
- if $\gamma = 1$, there exists $T_1 > 0$ such that, for every $T > T_1$, system (17) is observable in time T uniformly with respect to n .

The proof of these results relies on a new Carleman estimate for the solutions of (17).

Let $\gamma \in (0, 1]$, $T > 0$, and fix $n \in \mathbb{N}^*$ all over the proof. In order to simplify the notation, we write g and g_0 instead of g_n and $g_{0,n}$. Let a', b' be such that $a < a' < b' < b$.

First case: $\gamma \in [1/2, 1]$

In order to deduce the Carleman inequality, we define a weight function

$$\alpha(t, x) := \frac{M\beta(x)}{t(T-t)}, \quad (t, x) \in (0, T) \times \mathbb{R}, \quad (29)$$

where $\beta \in C^2(\mathbb{R}; \mathbb{R}_+)$ satisfies

$$\beta \geq 1 \text{ on } (-1, 1), \quad (30)$$

$$|\beta'| > 0 \text{ on } [-1, a'] \cup [b', 1], \quad (31)$$

$$\beta'(1) > 0, \quad \beta'(-1) < 0, \quad (32)$$

$$\beta'' < 0 \text{ on } [-1, a'] \cup [b', 1], \quad (33)$$

and $M > 0$ will be chosen later on. We also introduce the function

$$z(t, x) := g(t, x)e^{-\alpha(t, x)}, \quad (34)$$

that satisfies

$$P_1 z + P_2 z = P_3 z, \quad (35)$$

where

$$\begin{aligned} P_1 z &:= -\frac{\partial^2 z}{\partial x^2} + (\alpha_t - \alpha_x^2)z + (n\pi)^2 |x|^{2\gamma} z, & P_2 z &:= \frac{\partial z}{\partial t} - 2\alpha_x \frac{\partial z}{\partial x}, \\ P_3 z &:= \alpha_{xx} z. \end{aligned} \quad (36)$$

We develop the classical proof, taking the $L^2(Q)$ -norm in the identity (35), then developing the double product, which leads to

$$\int_Q P_1 z P_2 z \leq \frac{1}{2} \int_Q |P_3 z|^2, \quad (37)$$

where $Q := (0, T) \times (-1, 1)$ and we compute precisely each term.

Terms concerning $-\partial_x^2 z$: Integrating by parts, we get

$$-\int_Q \frac{\partial^2 z}{\partial x^2} \frac{\partial z}{\partial t} dx dt = \int_Q \frac{\partial z}{\partial x} \frac{\partial^2 z}{\partial t \partial x} dx dt = \int_0^T \frac{1}{2} \frac{d}{dt} \int_{-1}^1 \left| \frac{\partial z}{\partial x} \right|^2 dx dt = 0, \quad (38)$$

because $\partial_t z(t, \pm 1) = 0$ and $z(0) \equiv z(T) \equiv 0$, which is a consequence of assumptions (34), (29) and (30). Moreover,

$$\begin{aligned} \int_Q \frac{\partial^2 z}{\partial x^2} 2\alpha_x \frac{\partial z}{\partial x} dx dt &= - \int_Q \left| \frac{\partial z}{\partial x} \right|^2 \alpha_{xx} dx dt \\ &\quad + \int_0^T \left(\alpha_x(t, 1) \left| \frac{\partial z}{\partial x}(t, 1) \right|^2 - \alpha_x(t, -1) \left| \frac{\partial z}{\partial x}(t, -1) \right|^2 \right) dt. \end{aligned} \quad (39)$$

Terms concerning $(\alpha_t - \alpha_x^2)_x$: Again integrating by parts, we have

$$\int_Q (\alpha_t - \alpha_x^2)_x z \frac{\partial z}{\partial t} dx dt = - \frac{1}{2} \int_Q (\alpha_t - \alpha_x^2)_t |z|^2 dx dt. \quad (40)$$

Indeed, the boundary terms at $t = 0$ and $t = T$ vanish because, thanks to (34), (29), (30),

$$|(\alpha_t - \alpha_x^2)_t |z|^2| \leq \frac{1}{[t(T-t)]^2} e^{\frac{-M}{t(T-t)}} |M(T-2t)\beta + (M\beta')^2| \cdot |g|^2$$

tends to zero when $t \rightarrow 0$ and $t \rightarrow T$, for every $x \in [-1, 1]$. Moreover,

$$-2 \int_Q (\alpha_t - \alpha_x^2)_x z \alpha_x \frac{\partial z}{\partial x} dx dt = \int_Q [(\alpha_t - \alpha_x^2)_x]_x |z|^2 dx dt, \quad (41)$$

thanks to an integration by parts in the space variable.

Terms concerning $(n\pi)^2 |x|^{2\gamma} z$: First, since $z(0) \equiv z(T) \equiv 0$,

$$\int_Q (n\pi)^2 |x|^{2\gamma} z \frac{\partial z}{\partial t} dx dt = \frac{1}{2} \int_0^T \frac{d}{dt} \int_{-1}^1 (n\pi)^2 |x|^{2\gamma} |z|^2 dx dt = 0. \quad (42)$$

Furthermore, thanks to an integration by parts in the space variable,

$$-2 \int_Q (n\pi)^2 |x|^{2\gamma} z \alpha_x \frac{\partial z}{\partial x} dx dt = \int_Q [n^2 \pi^2 |x|^{2\gamma} \alpha_x]_x z^2 dx dt. \quad (43)$$

Combining (37), (38), (39), (40), (41), (42) and (43), we conclude that

$$\begin{aligned} \int_Q |z|^2 \left\{ -\frac{1}{2}(\alpha_t - \alpha_x^2)_t + [(\alpha_t - \alpha_x^2)_x]_x + n^2 \pi^2 [|x|^{2\gamma} \alpha_x]_x - \frac{1}{2} \alpha_{xx}^2 \right\} dx dt \\ + \int_0^T \left(\alpha_x(t, 1) \left| \frac{\partial z}{\partial x}(t, 1) \right|^2 - \alpha_x(t, -1) \left| \frac{\partial z}{\partial x}(t, -1) \right|^2 \right) dt \\ - \int_Q \left| \frac{\partial z}{\partial x} \right|^2 \alpha_{xx} dx dt \leq 0. \end{aligned} \quad (44)$$

In view of (32), we have $\alpha_x(t, 1) \geq 0$ and $\alpha_x(t, -1) \leq 0$, thus (44) yields

$$\begin{aligned} \int_Q |z|^2 \left\{ -\frac{1}{2}(\alpha_t - \alpha_x^2)_t + [(\alpha_t - \alpha_x^2)_x]_x - \frac{1}{2} \alpha_{xx}^2 + n^2 \pi^2 [|x|^{2\gamma} \alpha_x]_x \right\} dx dt \\ - \int_Q \left| \frac{\partial z}{\partial x} \right|^2 \alpha_{xx} dx dt \leq 0. \end{aligned} \quad (45)$$

Now, in the left hand side of (45) we separate the terms on $(0, T) \times (a', b')$ and those on $(0, T) \times [(-1, a') \cup (b', 1)]$. One has

$$\begin{aligned} -\alpha_{xx}(t, x) &\geq \frac{C_1 M}{t(T-t)} \quad \forall x \in [-1, a'] \cup [b', 1], \quad t \in (0, T), \\ |\alpha_{xx}(t, x)| &\leq \frac{C_2 M}{t(T-t)} \quad \forall x \in [a', b'], \quad t \in (0, T), \end{aligned} \quad (46)$$

where $C_1 := \min\{-\beta''(x); x \in [-1, a'] \cup [b', 1]\}$ is positive thanks to the assumption (33) and $C_2 := \sup\{|\beta''(x)|; x \in [a', b']\}$. Moreover,

$$\begin{aligned} -\frac{1}{2}(\alpha_t - \alpha_x^2)_t + [(\alpha_t - \alpha_x^2)\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2 &= \frac{1}{(t(T-t))^3} \left\{ M\beta(3Tt - T^2 - 3t^2) \right. \\ &\quad \left. + M^2 \left[(2t - T)\beta''\beta - \frac{t(T-t)\beta''^2}{2} \right] - 3M^3\beta''\beta'^2 \right\}. \end{aligned}$$

Hence, owing to (31) and (33), there exist $M_1 = M_1(T, \beta) > 0$, $C_3 = C_3(\beta) > 0$ and $C_4 = C_4(T, \beta) > 0$ such that, for every $M \geq M_1$ and $t \in (0, T)$,

$$\begin{aligned} -\frac{1}{2}(\alpha_t - \alpha_x^2)_t + [(\alpha_t - \alpha_x^2)\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2 &\geq \frac{C_3 M^3}{[t(T-t)]^3} \quad \forall x \in [-1, a'] \cup [b', 1], \\ \left| -\frac{1}{2}(\alpha_t - \alpha_x^2)_t + [(\alpha_t - \alpha_x^2)\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2 \right| &\leq \frac{C_4 M^3}{[t(T-t)]^3} \quad \forall x \in [a', b']. \end{aligned} \quad (47)$$

Using (45), (46) and (47), we deduce, for every $M \geq M_1$,

$$\begin{aligned} &\int_0^T \int_{(-1, a') \cup (b', 1)} \frac{C_1 M}{t(T-t)} \left| \frac{\partial z}{\partial x} \right|^2 dx dt \\ &\quad + \int_0^T \int_{(-1, a') \cup (b', 1)} \left[\frac{C_3 M^3}{(t(T-t))^3} |z|^2 + (n\pi)^2 [|x|^{2\gamma} \alpha_x]_x |z|^2 \right] dx dt \\ &\leq \int_0^T \int_{a'}^{b'} \left[\frac{C_2 M}{t(T-t)} \left| \frac{\partial z}{\partial x} \right|^2 + \frac{C_4 M^3}{(t(T-t))^3} |z|^2 - (n\pi)^2 [|x|^{2\gamma} \alpha_x]_x |z|^2 \right] dx dt. \end{aligned} \quad (48)$$

Moreover, for every $x \in (-1, 1)$, we have

$$|(n\pi)^2 [|x|^{2\gamma} \alpha_x]_x| = \frac{M(n\pi)^2}{t(T-t)} \left| 2\gamma \text{sign}(x) |x|^{2\gamma-1} \beta'(x) + |x|^{2\gamma} \beta''(x) \right| \leq \frac{C_5 n^2 M}{t(T-t)},$$

where $C_5 := \pi^2 \max\{2\gamma |x|^{2\gamma-1} |\beta'(x)| + |x|^{2\gamma} |\beta''(x)|; x \in [-1, 1]\}$ is finite because $2\gamma - 1 \geq 0$. From now on, we take

$$M = M(T, \beta, n) := \max\{1, M_1(T, \beta), M_2(T, \beta, n)\}, \quad (49)$$

where $M_2 = M_2(T, n)$ is defined by

$$M_2 := \sqrt{\frac{2C_5}{C_3}} n \left(\frac{T}{2} \right)^2. \quad (50)$$

Since

$$|(n\pi)^2[|x|^{2\gamma}\alpha_x]_x| \leq \frac{C_3 M^3}{2[t(T-t)]^3} \quad \forall (t, x) \in Q,$$

we conclude that

$$\begin{aligned} \int_0^T \int_{(-1, a') \cup (b', 1)} \frac{C_3 M^3}{2(t(T-t))^3} |z|^2 dx dt \\ \leq \int_0^T \int_{a'}^{b'} \frac{C_2 M}{t(T-t)} \left| \frac{\partial z}{\partial x} \right|^2 + \frac{C_6 M^3}{(t(T-t))^3} |z|^2 dx dt, \end{aligned} \quad (51)$$

where $C_6 = C_6(T, \beta) := C_4 + C_3/2$. Coming back to our original variables thanks to identity (34), we have

$$\begin{aligned} \int_0^T \int_{(-1, a') \cup (b', 1)} \frac{C_3 M^3 |g|^2 e^{-2\alpha}}{2(t(T-t))^3} dx dt \\ \leq \int_0^T \int_{a'}^{b'} \left(\frac{C_7 M^3 |g|^2}{(t(T-t))^3} + \frac{C_8 M}{t(T-t)} \left| \frac{\partial g}{\partial x} \right|^2 \right) e^{-2\alpha} dx dt, \end{aligned} \quad (52)$$

where $C_8 = C_8(T, \beta) := 2C_2$ and $C_7 = C_7(T, \beta) := C_6 + 2C_2 \sup\{\beta'(x)^2; x \in [a', b']\}$. Owing to (30) and the assumption $M \geq 1$, we have, for every $x \in [a, b]$, $t \in (0, T)$,

$$\begin{aligned} \frac{C_7 M^3}{(t(T-t))^3} e^{-2\alpha} &\leq \frac{C_7 M^3}{(t(T-t))^3} e^{-\frac{2M}{t(T-t)}} \leq C_9, \\ \frac{C_8 M}{t(T-t)} e^{-2\alpha} &\leq \frac{C_8 t(T-t)}{M} \left(\frac{M}{t(T-t)} \right)^2 e^{-\frac{2M}{t(T-t)}} \leq C_{10} t(T-t), \end{aligned}$$

where $C_9 = C_9(T, \beta) := C_7 \sup\{x^3 e^{-2x}; x \in \mathbb{R}_+\}$ and $C_{10} = C_{10}(T, \beta) := C_8 \sup\{x^2 e^{-2x}; x \in \mathbb{R}_+\}$. Therefore, from (52) we deduce

$$\begin{aligned} \int_0^T \int_{(-1, a') \cup (b', 1)} \frac{C_3 M^3 |g|^2 e^{-2\alpha}}{2(t(T-t))^3} dx dt \\ \leq \int_0^T \int_{a'}^{b'} \left(C_9 |g|^2 + C_{10} t(T-t) \left| \frac{\partial g}{\partial x} \right|^2 \right) dx dt. \end{aligned} \quad (53)$$

Now, let us prove that the right hand side of the previous inequality can be bounded by a first order term in g on $(0, T) \times (a, b)$. We consider $\rho \in C^\infty(\mathbb{R}, \mathbb{R}_+)$ such that $0 \leq \rho \leq 1$,

$$\rho \equiv 1 \text{ on } (a', b'), \quad (54)$$

$$\rho \equiv 0 \text{ on } (-1, a) \cup (b, 1). \quad (55)$$

Multiplying the first equation of (17) by $g\rho t(T-t)$ and then integrating over $(0, T) \times (-1, 1)$, we get

$$\int_0^T \int_{-1}^1 \rho t(T-t) \left[\frac{1}{2} \frac{d}{dt} (|g|^2) - \frac{\partial^2 g}{\partial x^2} g + (n\pi)^2 |x|^{2\gamma} |g|^2 \right] dx dt = 0. \quad (56)$$

Integrating by parts with respect to space and time, we obtain

$$\frac{1}{2} \int_0^T \int_{-1}^1 \frac{d}{dt} [|g|^2] \rho t (T-t) dx dt = -\frac{1}{2} \int_0^T \int_{-1}^1 |g|^2 \rho (T-2t) dx dt, \quad (57)$$

$$\begin{aligned} & - \int_0^T \int_{-1}^1 \frac{\partial^2 g}{\partial x^2} g \rho t (T-t) dx dt \\ & = \int_0^T \left[\int_{-1}^1 \left| \frac{\partial g}{\partial x} \right|^2 \rho t (T-t) - \frac{1}{2} |g|^2 \rho'' t (T-t) \right] dx dt. \end{aligned} \quad (58)$$

Indeed, the boundary terms at $t = 0$ and $t = T$ in (57) vanish owing to the factor $t(T-t)$, and the boundary terms at $x = \pm 1$ in (58) vanish thanks to the boundary conditions on g . Combining (56), (57) and (58), we deduce

$$\begin{aligned} & \int_0^T \int_{-1}^1 t(T-t) \left(\left| \frac{\partial g}{\partial x} \right|^2 \rho - \frac{1}{2} |g|^2 \rho'' + (n\pi)^2 |x|^{2\gamma} |g|^2 \rho \right) dx dt \\ & - \frac{1}{2} \int_0^T \int_{-1}^1 |g|^2 \rho (T-2t) dx dt = 0. \end{aligned} \quad (59)$$

In view of (54), (55) and (59), we have

$$\begin{aligned} & \int_0^T \int_{a'}^{b'} \left| \frac{\partial g}{\partial x} \right|^2 t(T-t) dx dt \leq \int_0^T \int_{-1}^1 \left| \frac{\partial g}{\partial x} \right|^2 \rho t(T-t) dx dt \\ & = \frac{1}{2} \int_0^T \int_{-1}^1 |g|^2 [\rho(T-2t) + t(T-t)(\rho'' - 2(n\pi)^2 |x|^{2\gamma} \rho)] dx dt \\ & \leq C_{11} \int_0^T \int_a^b |g|^2 dx dt, \end{aligned} \quad (60)$$

where $C_{11} = C_{11}(T, \rho) := T \|\rho\|_{L^\infty} + \frac{T^2}{2} \|\rho''\|_{L^\infty}$. Combining inequalities (60) and (53) leads to

$$\int_0^T \int_{(-1, a') \cup (b', 1)} \frac{C_3 M^3 |g|^2 e^{-2\alpha}}{(t(T-t))^3} dx dt \leq \int_0^T \int_a^b C_{12} |g|^2 dx dt, \quad (61)$$

where $C_{12} = C_{12}(T, \beta, \rho) := 2[C_9 + C_{10}C_{11}]$. Since

$$\frac{2T^2}{9} \leq t(T-t) \leq \frac{T^2}{4} \quad \forall t \in \left[\frac{T}{3}, \frac{2T}{3} \right],$$

we have

$$\frac{e^{-2\alpha(t,x)}}{(t(T-t))^3} \geq \frac{e^{-9c_3 M/T^2}}{(T^2/4)^3} \quad \forall x \in [-1, a'] \cup [b', 1], \quad t \in (T/3, 2T/3),$$

where $c_3 = c_3(\beta) := \sup\{\beta(x); x \in [-1, a'] \cup [b', 1]\}$. Therefore, (61) implies

$$\frac{C_3 M^3}{(T^2/4)^3} e^{-\frac{9c_3 M}{T^2}} \int_{T/3}^{2T/3} \int_{(-1, a') \cup (b', 1)} |g|^2 dx dt \leq \int_0^T \int_a^b C_{12} |g|^2 dx dt. \quad (62)$$

Adding the same quantity to both sides and using the inclusion $(a', b') \subset (a, b)$, we obtain

$$\begin{aligned} \frac{C_3 M^3}{(T^2/4)^3} e^{-\frac{9c_3 M}{T^2}} \int_{T/3}^{2T/3} \int_{-1}^1 |g|^2 dx dt \\ \leq \left(C_{12} + \frac{C_3 M^3}{(T^2/4)^3} e^{-\frac{9c_3 M}{T^2}} \right) \int_0^T \int_a^b |g|^2 dx dt, \end{aligned}$$

which can also be written as

$$\int_{T/3}^{2T/3} \int_{-1}^1 |g|^2 dx dt \leq \left(C_{12} \frac{(T^2/4)^3}{C_3 M^3} e^{\frac{9c_3 M}{T^2}} + 1 \right) \int_0^T \int_a^b |g|^2 dx dt. \quad (63)$$

Now, thanks to Proposition 3,

$$\int_{-1}^1 |g(T, x)|^2 dx \leq e^{-\frac{T}{3} c_* n^{\frac{2}{1+\gamma}}} \int_{-1}^1 |g(t, x)|^2 dx \quad \forall t \in [T/3, 2T/3].$$

Thus,

$$\int_{-1}^1 |g(T, x)|^2 dx \leq \frac{3}{T} e^{-\frac{T}{3} c_* n^{\frac{2}{1+\gamma}}} \left(C_{12} \frac{(T^2/4)^3}{C_3 M^3} e^{\frac{9c_3 M}{T^2}} + 1 \right) \int_0^T \int_a^b |g|^2 dx dt. \quad (64)$$

Now, let

$$\mathcal{C} = \mathcal{C}(\beta) := \frac{1}{2} \sqrt{\frac{C_5}{2C_3}}.$$

Then, there exists $n_1 = n_1(T, \beta) \in \mathbb{N}^*$ such that, for every $n \geq n_1$, the quantity $M = M(T, \beta, n)$ defined by (49) satisfies

$$M = M(T, \beta, n) = \mathcal{C} n T^2.$$

Hence, for every $n \geq n_1$,

$$-c_* n^{\frac{2}{1+\gamma}} \frac{T}{3} + \frac{9c_3 M}{T^2} = -c_* n^{\frac{2}{1+\gamma}} \frac{T}{3} + 9c_3 \mathcal{C} n,$$

where c_* , c_3 , \mathcal{C} depend only on β and γ .

Let us assume that $\gamma \in [1/2, 1)$ and $T > 0$ is arbitrary. Since $2/(1+\gamma) > 1$, there exists $n_2 = n_2(\beta) \in \mathbb{N}^*$ such that

$$-c_* n^{\frac{2}{1+\gamma}} \frac{T}{3} + 9c_3 \mathcal{C} n \leq 0 \quad \forall n \geq n_2.$$

So, inequality (64) yields, for every $n \geq \max\{n_1, n_2\}$,

$$\int_{-1}^1 |g(T, x)|^2 dx \leq \frac{3}{T} \left(C_{12} \frac{(T^2/4)^3}{C_3 M^3} + 1 \right) \int_0^T \int_a^b |g|^2 dx dt, \quad (65)$$

which in turn implies the conclusion.

Next, let us assume that $\gamma = 1$ and $T > T_\sharp$, where

$$T_\sharp = T_\sharp(\beta) := \frac{27c_3 \mathcal{C}}{c_*}.$$

Then, once again we recover (65) for every $n \geq n_1$, and the conclusion follows as above.

Second case: $\gamma \in (0, 1/2)$.

The previous strategy does not apply to $\gamma \in (0, 1/2)$ because the term $(n\pi)^2[|x|^{2\gamma}\alpha_x]_x$ (that diverges at $x = 0$) in (48) can no longer be bounded by $\frac{C_3M^3}{(t(T-t))^3}$ (which is bounded at $x = 0$). Note that both terms are of the same order as M^3 , because of the dependence of M with respect to n in (49). In order to deal with this difficulty, we adapt the choice of the weight β and the dependence of M with respect to n .

Let β be a C^1 -function on $(-1, 1)$, which is also C^2 on $[-1, 0)$ and $(0, 1]$, but such that β'' diverges at zero. More precisely, we assume that assumptions (30), (31) and (32) hold, and condition (33) is replaced by

$$\beta'' < 0 \text{ on } [-1, 0) \cup (0, a'] \cup [b', 1]. \quad (66)$$

Moreover, β has the following form on a neighborhood $(-\epsilon, \epsilon)$ of 0

$$\beta(x) = \mathcal{C}_0 - \int_0^x \sqrt{\text{sign}(s)|s|^{2\gamma} + \mathcal{C}_1} ds \quad \forall x \in (-\epsilon, \epsilon), \quad (67)$$

where the constant $\mathcal{C}_0, \mathcal{C}_1$ are large enough so that $\beta \geq 1$ and $\beta' < 0$ on $(-\epsilon, \epsilon)$, respectively. Then,

$$\beta'(x) = -\sqrt{\text{sign}(x)|x|^{2\gamma} + \mathcal{C}_1} \quad \forall x \in (-\epsilon, \epsilon), \quad (68)$$

thus β'' diverges at $x = 0$.

Performing the same computations as in the previous case, we get to inequality (45). Then, owing to (31) and (66), there exist $M_1 = M_1(T, \beta) > 0$, $C_3 = C_3(\beta) > 0$ and $C_4 = C_4(T, \beta) > 0$ such that, for every $M \geq M_1$ and $t \in (0, T)$,

$$\begin{aligned} & -\frac{1}{2}(\alpha_t - \alpha_x^2)_t + [(\alpha_t - \alpha_x^2)\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2 \\ & \geq \frac{C_3M^3}{[t(T-t)]^3} |\beta''(x)| \beta'(x)^2 \quad \forall x \in [-1, 0) \cup (0, a'] \cup [b', 1], \\ & \left| -\frac{1}{2}(\alpha_t - \alpha_x^2)_t + [(\alpha_t - \alpha_x^2)\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2 \right| \leq \frac{C_4M^3}{[t(T-t)]^3} \quad \forall x \in [a', b']. \end{aligned}$$

In view of (32), for every $M \geq M_1$,

$$\begin{aligned} & \int_0^T \int_{(-1, a') \cup (b', 1)} \left[\frac{C_3M^3}{(t(T-t))^3} |\beta''(x)| \beta'(x)^2 |z|^2 + (n\pi)^2 [|x|^{2\gamma}\alpha_x]_x |z|^2 \right] dx dt \\ & \leq \int_0^T \int_{a'}^{b'} \left[\frac{C_2M}{t(T-t)} \left| \frac{\partial z}{\partial x} \right|^2 + \frac{C_4M^3}{(t(T-t))^3} |z|^2 - (n\pi)^2 [|x|^{2\gamma}\alpha_x]_x |z|^2 \right] dx dt. \end{aligned} \quad (69)$$

Moreover,

$$\begin{aligned} & |(n\pi)^2 [|x|^{2\gamma}\alpha_x]_x| = (n\pi)^2 \frac{M}{t(T-t)} \left| 2\gamma \text{sign}(x) |x|^{2\gamma-1} \beta'(x) + |x|^{2\gamma} \beta''(x) \right| \\ & \leq \frac{C_5 n^2 M}{t(T-t)} \left(|x|^{2\gamma-1} |\beta'(x)| + |x|^{2\gamma} |\beta''(x)| \right) \quad \forall x \in (-1, 0) \cup (0, 1), \end{aligned}$$

where $C_5 = \pi^2(2\gamma + 1)$. From now on, we take

$$M = M(T, \beta, n) := \max \left\{ 1; M_1(T, \beta); \frac{nT^2}{\lambda} \right\}, \quad (70)$$

where $\lambda > 0$ is a (small enough) constant, that will be chosen later on.

Then, there exists $n_1 = n_1(T, \lambda, \beta) \in \mathbb{N}^*$ such that, for every $n \geq n_1$, we have $M = nT^2/\lambda$. Therefore, for every $x \in (-1, 0) \cup (0, 1)$,

$$|(n\pi)^2[|x|^{2\gamma}\alpha_x]_x| \leq \frac{C_6\lambda^2M^3}{(t(T-t))^3} \left(|x|^{2\gamma-1}|\beta'(x)| + |x|^{2\gamma}|\beta''(x)| \right),$$

where $C_6 = C_6(\gamma) > 0$. Let us verify that, for $\lambda > 0$ small enough and for every $x \in (-1, 0) \cup (0, a') \cup (b', 1)$, we have

$$\begin{aligned} \frac{C_6\lambda^2M^3}{(t(T-t))^3} |x|^{2\gamma-1}|\beta'(x)| &\leq \frac{C_3M^3}{4(t(T-t))^3} |\beta''(x)| |\beta'(x)|^2, \\ \frac{C_6\lambda^2M^3}{(t(T-t))^3} |x|^{2\gamma}|\beta''(x)| &\leq \frac{C_3M^3}{4(t(T-t))^3} |\beta''(x)| |\beta'(x)|^2, \end{aligned}$$

or, equivalently, for every $x \in (-1, 0) \cup (0, a') \cup (b', 1)$,

$$\begin{aligned} C_6\lambda^2|x|^{2\gamma-1} &\leq \frac{C_3}{4} |\beta''(x)| \cdot |\beta'(x)|, \\ C_6\lambda^2|x|^{2\gamma} &\leq \frac{C_3}{4} \beta'(x)^2. \end{aligned} \quad (71)$$

The second inequality is easy to satisfy (for $\lambda = \lambda(\beta, \gamma)$ small enough), because $|\beta'| > 0$ on $[-1, a'] \cup [b', 1]$. Thanks to (68), for every $x \in (-\epsilon, \epsilon)$,

$$\beta'(x)^2 = \text{sign}(x)|x|^{2\gamma} + C_1,$$

so

$$\beta''(x)\beta'(x) = \gamma|x|^{2\gamma-1}.$$

Therefore, for every $x \in (-\epsilon, \epsilon) \setminus \{0\}$, the first inequality in (71) is equivalent to

$$C_6\lambda^2 \leq \frac{C_3}{4} \gamma,$$

which is trivially satisfied, when $\lambda = \lambda(\beta)$ is small enough. Moreover, the first inequality of (71) holds for every $x \in [-1, -\epsilon] \cup [\epsilon, a'] \cup [b', 1]$ when λ is small enough, since $|\beta''\beta'| > 0$ on this compact set. Finally, we deduce

$$\begin{aligned} \int_0^T \int_{(-1, a') \cup (b', 1)} \frac{C_3M^3}{2(t(T-t))^3} |\beta''(x)| |\beta'(x)|^2 |z|^2 dx dt \\ \leq \int_0^T \int_{a'}^{b'} \left[\frac{C_2M}{t(T-t)} \left| \frac{\partial z}{\partial x} \right|^2 + \frac{C_4M^3}{(t(T-t))^3} |z|^2 \right] dx dt, \end{aligned} \quad (72)$$

where $C_4 = C_4(T, \beta) > 0$. Since the function $|\beta''|(\beta')^2$ is bounded from below by some positive constant on $[-1, a'] \cup [b', 1]$, we also have

$$\begin{aligned} \int_0^T \int_{(-1, a') \cup (b', 1)} \frac{C'_3M^3}{2(t(T-t))^3} |z|^2 dx dt \\ \leq \int_0^T \int_{a'}^{b'} \left[\frac{C_2M}{t(T-t)} \left| \frac{\partial z}{\partial x} \right|^2 + \frac{C_4M^3}{(t(T-t))^3} |z|^2 \right] dx dt, \end{aligned} \quad (73)$$

and the proof may be finished in the same way as in the first case. \square

2.6 Proof of the negative statements of Theorem 5

The goal of this section is the proof of the following results:

- if $\gamma = 1$, then there exists $T_2 > 0$ such that, for every $T < T_2$, system (17) is not observable in time T uniformly with respect to n ;
- if $\gamma > 1$ and $T > 0$, then system (17) is not observable in time T uniformly with respect to n .

The proof relies on the choice of particular test functions, that falsify uniform observability.

Let $\gamma \in [1, +\infty)$ be fixed and $T > 0$. For every $n \in \mathbb{N}^*$, we denote by λ_n (instead of $\lambda_{n,\gamma}$) the first eigenvalue of the operator $A_{n,\gamma}$ defined in Section 2.4, and by v_n the associated positive eigenvector of norm one, that is,

$$\begin{cases} -v_n''(x) + [(n\pi)^2|x|^{2\gamma} - \lambda_n]v_n(x) = 0, & x \in (-1, 1), \quad n \in \mathbb{N}^*, \\ v_n(\pm 1) = 0, \quad v_n \geq 0, \\ \|v_n\|_{L^2(-1,1)} = 1. \end{cases}$$

Then, for every $n \geq 1$, the function

$$g_n(t, x) := v_n(x)e^{-\lambda_n t} \quad \forall (t, x) \in \mathbb{R} \times (-1, 1),$$

solves the adjoint system (17). Let us note that

$$\begin{aligned} \int_{-1}^1 g_n(T, x)^2 dx &= e^{-2\lambda_n T}, \\ \int_0^T \int_a^b g_n(t, x)^2 dx dt &= \frac{1 - e^{-2\lambda_n T}}{2\lambda_n} \int_a^b v_n(x)^2 dx. \end{aligned}$$

So, in order to prove that uniform observability fails, it suffices to show that

$$\frac{e^{2\lambda_n T}}{\lambda_n} \int_a^b v_n(x)^2 dx \rightarrow 0 \text{ when } n \rightarrow +\infty. \quad (74)$$

In order to estimate the last integral, we will compare v_n with an explicit supersolution of the problem on a suitable subinterval of $[-1, 1]$.

Lemma 3. *Let $0 < a < b < 1$. For every $n \in \mathbb{N}^*$, set*

$$x_n := \left(\frac{\lambda_n}{(n\pi)^2} \right)^{\frac{1}{2\gamma}} \quad (75)$$

and let $W_n \in C^2([x_n, 1], \mathbb{R})$ be a solution of

$$\begin{cases} -W_n''(x) + [(n\pi)^2 x^{2\gamma} - \lambda_n]W_n(x) \geq 0, & x \in (x_n, 1), \\ W_n(1) \geq 0, \\ W_n'(x_n) < -\sqrt{x_n} \lambda_n. \end{cases} \quad (76)$$

Then there exists $n_* \in \mathbb{N}^*$ such that, for every $n \geq n_*$,

$$\int_a^b v_n(x)^2 dx \leq \int_a^b W_n(x)^2 dx.$$

Proof: First, let us observe that, thanks to the second statement of Proposition 3, $x_n \rightarrow 0$ when $n \rightarrow +\infty$. In particular, there exists $n_* \geq 1$ such that $x_n \leq a$ for every $n \geq n_*$. Now, let us prove that $|v'_n(x_n)| \leq \sqrt{x_n} \lambda_n$ for all $n \geq n_*$. Indeed, from Lemma 2, we have $v_n(x) = v_n(-x)$, thus $v'_n(0) = 0$. Hence, thanks to the Cauchy-Schwarz inequality and the relation $\|v_n\|_{L^2(-1,1)} = 1$,

$$\begin{aligned} |v'_n(x_n)| &= \left| \int_0^{x_n} v''_n(s) ds \right| = \left| \int_0^{x_n} [(n\pi)^2 |s|^{2\gamma} - \lambda_n] v_n(s) ds \right| \\ &\leq \left(\int_0^{x_n} [(n\pi)^2 |s|^{2\gamma} - \lambda_n]^2 ds \right)^{1/2} \left(\int_0^{x_n} v_n(s)^2 ds \right)^{1/2} \leq \sqrt{x_n} \lambda_n. \end{aligned}$$

Furthermore, we claim that $v_n(x) \leq W_n(x)$ for every $x \in [x_n, 1]$, $n \geq n_*$. Indeed, if not so, there exists $x_* \in [x_n, 1]$ such that

$$(W_n - v_n)(x_*) = \min\{(W_n - v_n)(x); x \in [x_n, 1]\} < 0.$$

Since $(W_n - v_n)(1) \geq 0$ and $(W_n - v_n)'(x_n) < 0$, we have $x_* \in (x_n, 1)$. Moreover, the function $W_n - v_n$ has a minimum at x_* , thus $(W_n - v_n)'(x_*) = 0$ and $(W_n - v_n)''(x_*) \geq 0$. Therefore,

$$-(W_n - v_n)''(x_*) + [(n\pi)^2 |x_*|^{2\gamma} - \lambda_n](W_n - v_n)(x_*) < 0,$$

which is a contradiction. Our claim follows and the proof is complete. \square

In order to apply Lemma 3, we look for an explicit supersolution W_n of (76), of the form

$$W_n(x) = C_n e^{-\mu_n x^{\gamma+1}}, \quad (77)$$

where $C_n, \mu_n > 0$. Thus, the condition $W_n(1) \geq 0$ is automatically satisfied.

First step: Let us prove that, for an appropriate choice of μ_n , the first inequality of (76) holds. Since

$$W'_n(x) = -\mu_n(\gamma + 1)x^\gamma W_n(x),$$

$$W''_n(x) = [-\mu_n\gamma(\gamma + 1)x^{\gamma-1} + \mu_n^2(\gamma + 1)^2 x^{2\gamma}]W_n(x),$$

the first inequality of (76) holds if and only if, for every $x \in (x_n, 1)$,

$$[(n\pi)^2 - \mu_n^2(\gamma + 1)^2]x^{2\gamma} + \mu_n\gamma(\gamma + 1)x^{\gamma-1} \geq \lambda_n. \quad (78)$$

In particular, it holds when

$$\mu_n \leq \frac{n\pi}{\gamma + 1} \quad (79)$$

and

$$[(n\pi)^2 - \mu_n^2(\gamma + 1)^2]x_n^{2\gamma} + \mu_n\gamma(\gamma + 1)x_n^{\gamma-1} \geq \lambda_n. \quad (80)$$

Indeed, in this case, the left hand side of (78) is an increasing function of x . In view of (75), and after several simplifications, inequality (80) can be recast as

$$\mu_n \leq \frac{\gamma}{\gamma + 1} \left(\frac{(n\pi)^2}{\lambda_n} \right)^{\frac{1}{2} + \frac{1}{2\gamma}}.$$

So, recalling (79), in order to satisfy the first inequality of (76) we can take

$$\mu_n := \min \left\{ \frac{n\pi}{\gamma+1}; \frac{\gamma}{\gamma+1} \left(\frac{(n\pi)^2}{\lambda_n} \right)^{\frac{1}{2} + \frac{1}{2\gamma}} \right\}. \quad (81)$$

For the following computations, it is important to notice that, thanks to (81) and the second statement of Proposition 3, for n large enough μ_n is of the form

$$\mu_n = C_1(\gamma)n. \quad (82)$$

Second step: Let us prove that, for an appropriate choice of C_n , the third inequality of (76) holds. Since

$$W'_n(x_n) = -C_n \mu_n (\gamma+1) x_n^\gamma e^{-\mu_n x_n^{\gamma+1}},$$

the third inequality of (76) is equivalent to

$$C_n > \frac{\lambda_n e^{\mu_n x_n^{\gamma+1}}}{(\gamma+1) \mu_n x_n^{\gamma-\frac{1}{2}}}.$$

Therefore, it is sufficient to choose

$$C_n := \frac{2\lambda_n e^{\mu_n x_n^{\gamma+1}}}{(\gamma+1) \mu_n x_n^{\gamma-\frac{1}{2}}}. \quad (83)$$

Third step: Let us prove condition (74). Thanks to Lemma 3, (77), (82) and (83), for every $n \geq n_*$,

$$\begin{aligned} \frac{e^{2\lambda_n T}}{\lambda_n} \int_a^b v_n(x)^2 dx &\leq \frac{e^{2\lambda_n T}}{\lambda_n} \int_a^b W_n(x)^2 dx \leq \frac{e^{2\lambda_n T}}{\lambda_n} W_n(a)^2 \\ &\leq \frac{e^{2\lambda_n T}}{\lambda_n} C_n^2 e^{-2\mu_n a^{1+\gamma}} \leq \frac{e^{2\lambda_n T}}{\lambda_n} \frac{4\lambda_n^2 e^{2\mu_n x_n^{\gamma+1}}}{(\gamma+1)^2 \mu_n^2 x_n^{2\gamma-1}} e^{-2\mu_n a^{1+\gamma}}. \end{aligned}$$

By identities (75), (82) and Proposition 3, we have

$$\mu_n x_n^{\gamma+1} \leq C_2(\gamma) \quad \forall n \in \mathbb{N}^*,$$

thus

$$\frac{e^{2\lambda_n T}}{\lambda_n} \int_a^b v_n(x)^2 dx \leq e^{2n(\frac{\lambda_n}{n}T - C_1(\gamma)a^{1+\gamma})} \frac{4\lambda_n e^{2C_2(\gamma)}}{(\gamma+1)^2 \mu_n^2 x_n^{2\gamma-1}}. \quad (84)$$

If $\gamma > 1$, we deduce from the second statement of Proposition 3 that

$$\frac{\lambda_n}{n} \rightarrow 0 \quad \text{when } n \rightarrow +\infty.$$

So, for every $T > 0$, there exists $n_\# \geq n_*$ such that, for every $n \geq n_\#$,

$$\frac{\lambda_n}{n} T - C_1(\gamma) a^{1+\gamma} < -\frac{1}{2} C_1(\gamma) a^{1+\gamma}. \quad (85)$$

Then, inequality (84) proves condition (74) (since the term that multiplies the exponential behaves like a rational fraction of n).

If $\gamma = 1$, Proposition 3 ensures that $c_* n \leq \lambda_n \leq c^* n$, thus we deduce (85), hence (74), for every

$$T < \frac{C_1(\gamma) a^{1+\gamma}}{2c_*}.$$

2.7 End of the proof of Theorem 2

The first (resp. third) statement of Theorem 2 has been proved in Subsection 2.5 (resp. 2.6); let us prove the second one.

Let us consider $\gamma = 1$. Thanks to the results of Subsection 2.5, the quantity

$$T^* := \inf\{T > 0; \text{ system (17) is observable in time } T \text{ uniformly in } n\}$$

is well defined in $[0, +\infty)$. Moreover, as showed in Subsection 2.6, $T^* > 0$. Clearly, uniform observability in some time T_\sharp implies uniform observability in any time $T > T_\sharp$, so

- for every $T > T^*$, system (17) is observable in time T uniformly with respect to n ;
- for every $T < T^*$, system (17) is not observable in time T uniformly with respect to n .

3 Conclusion and open problems

In this article we have studied the null controllability of the Grushin type equation (1), in a rectangle, with a distributed control localized on a strip parallel to the y -axis. We have proved that null controllability

- holds in any positive time, when degeneracy is not too strong, i.e. $\gamma \in (0, 1)$,
- holds only in large time, when $\gamma = 1$,
- does not hold when degeneracy is too strong, i.e. $\gamma > 1$.

Null controllability when $\gamma \in (0, 1]$ and the control region ω is more general is an open problem. When $\gamma = 1$, it would be interesting to characterize the minimal time T^* required for null controllability, and possibly connect it with the associated diffusion process. Generalizations of this result to multidimensional configurations ($x \in (-1, 1)^m$, $y \in (0, 1)^n$), or boundary controls, are also open.

References

- [1] F. Alabau-Boussouira, P. Cannarsa, and G. Fragnelli. Carleman estimates for degenerate parabolic operators with applications to null controllability. *J. Evol. Equ.*, 6(2):161–204, 2006.
- [2] J.M. Ball. Strongly continuous semigroups, weak solutions, and the variation of constants formula. *Proc. Amer. Math. Soc.*, 63:370–373, July 1977.
- [3] K. Beauchard and E. Zuazua. Some controllability results for the 2D Kolmogorov equation. *Ann. Inst. H. Poincaré An. Non linéaire*, 26:1793–1815, 2009.
- [4] P. Cannarsa and L. de Teresa. Controllability of 1-d coupled degenerate parabolic equations. *Electron. J. Differ. Equ.*, Paper No. 73:21 p., 2009.

- [5] P. Cannarsa, G. Fragnelli, and D. Rocchetti. Null controllability of degenerate parabolic operators with drift. *Netw. Heterog. Media*, 2(4):695–715 (electronic), 2007.
- [6] P. Cannarsa, G. Fragnelli, and D. Rocchetti. Controllability results for a class of one-dimensional degenerate parabolic problems in nondivergence form. *J. evol. equ.*, 8:583–616, 2008.
- [7] P. Cannarsa, P. Martinez, and J. Vancostenoble. Persistent regional null controllability for a class of degenerate parabolic equations. *Commun. Pure Appl. Anal.*, 3(4):607–635, 2004.
- [8] P. Cannarsa, P. Martinez, and J. Vancostenoble. Null controllability of degenerate heat equations. *Adv. Differential Equations*, 10(2):153–190, 2005.
- [9] P. Cannarsa, P. Martinez, and J. Vancostenoble. Carleman estimates for a class of degenerate parabolic operators. *SIAM J. Control Optim.*, 47(1):1–19, 2008.
- [10] P. Cannarsa, P. Martinez, and J. Vancostenoble. Carleman estimates and null controllability for boundary-degenerate parabolic operators. *C. R. Math. Acad. Sci. Paris*, 347(3-4):147–152, 2009.
- [11] J.-M. Coron and S. Guerrero. Singular optimal control : a linear 1D parabolic-hyperbolic example. *Asymptotic Analysis*, 44 (3,4):237–257, 2005.
- [12] A. Doubova, E. Fernández-Cara, and E. Zuazua. On the controllability of parabolic systems with a nonlinear term involving the state and the gradient. *SIAM J. Control Optim.*, 42 (3):798–819, 2002.
- [13] T. Duyckaerts, X. Zhang, and E. Zuazua. On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials. *Ann. Inst. H. Poincaré : Analyse Nonlinéaire*, 25, 141, 2008.
- [14] C. Fabre, J.P. Puel, and E. Zuazua. Approximate controllability of the semilinear heat equation. *Proc. Roy. Soc. Edinburgh*, 125A:31–61, 1995.
- [15] H.O. Fattorini and D. Russel. Exact controllability theorems for linear parabolic equations in one space dimension. *Arch. Rational Mech. Anal.*, 43:272–292, 1971.
- [16] E. Fernández-Cara and E. Zuazua. Null and approximate controllability for weakly blowing up semilinear heat equations. *Annales de l’IHP. Analyse non linéaire*, 17:583–616, 2000.
- [17] E. Fernández-Cara and E. Zuazua. The cost of approximate controllability for heat equations: The linear case. *Advances in Differential Equations*, 5(4-6):465–514, 2000.
- [18] E. Fernández-Cara and E. Zuazua. On the null controllability of the one-dimensional heat equation with BV coefficients. *Computational and Applied Mathematics*, 12:167–190, 2002.

- [19] C. Flores and L. de Teresa. Carleman estimates for degenerate parabolic equations with first order terms and applications. *C. R. Math. Acad. Sci. Paris*, 348(7-8):391–396, 2010.
- [20] A.V. Fursikov and O.Y. Imanuvilov. Controllability of evolution equations. *Lecture Notes Series, Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul*, 34, 1996.
- [21] N. Garofalo and D. Vassilev. Strong unique continuation for generalized Baouendi-Grushin operators. In *Advances in analysis*, pages 255–263. World Sci. Publ., Hackensack, NJ, 2005.
- [22] M. González-Burgos and L. de Teresa. Some results on controllability for linear and nonlinear heat equations in unbounded domains. *Adv. Differential Equations*, 12 (11):1201–1240, 2007.
- [23] L. Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.
- [24] O.Y. Imanuvilov. Boundary controllability of parabolic equations. *Uspekhi. Mat. Nauk*, 48(3(291)):211–212, 1993.
- [25] O.Y. Imanuvilov. Controllability of parabolic equations. *Mat. Sb.*, 186(6):109–132, 1995.
- [26] O.Y. Imanuvilov and M. Yamamoto. Carleman estimate for a parabolic equation in sobolev spaces of negative order and its applications. *Control of Nonlinear Distributed Parameter Systems, G. Chen et al. eds., Marcel-Dekker*, pages 113–137, 2000.
- [27] G. Lebeau and L. Robbiano. Contrôle exact de l’équation de la chaleur. *Comm. P.D.E.*, 20:335–356, 1995.
- [28] J.-L. Lions. *Équations différentielles opérationnelles et problèmes aux limites*. Die Grundlehren der mathematischen Wissenschaften, Bd. 111. Springer-Verlag, Berlin, 1961.
- [29] J.-L. Lions. *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*. Avant propos de P. Lelong. Dunod, Paris, 1968.
- [30] A. Lopez and E. Zuazua. Uniform null controllability for the one dimensional heat equation with rapidly oscillating periodic density. *Annales IHP. Analyse non linéaire*, 19 (5):543–580, 2002.
- [31] P. Martinez and J. Vancostenoble. Carleman estimates for one-dimensional degenerate heat equations. *J. Evol. Equ.*, 6(2):325–362, 2006.
- [32] P. Martinez, J. Vancostenoble, and J.-P. Raymond. Regional null controllability of a linearized Crocco type equation. *SIAM J. Control Optim.*, 42, no. 2:709–728, 2003.
- [33] L. Miller. On the null-controllability of the heat equation in unbounded domains. *Bulletin des Sciences Mathématiques*, 129, 2:175–185, 2005.

- [34] L. Miller. On exponential observability estimates for the heat semigroup with explicit rates. *Rendiconti Lincei: Matematica e Applicazioni*, 17, 4:351–366, 2006.
- [35] Zabczyk. *Mathematical control theory: an introduction*. Birkhäuser, 2000.
- [36] E. Zuazua. Approximate controllability of the semilinear heat equation: boundary control. *International Conference in honour of Prof. R. Glowinski, Computational Sciences for the 21st Century, M.O. Bristeau et al. eds., John Wiley and Sons*, pages 738–747, 1997.
- [37] E. Zuazua. Finite dimensional null-controllability of the semilinear heat equation. *J.Math. Pures et Appl.*, 76:237–264, 1997.